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STABILITY OF ORTHOTROPIC VISCOELASTIC SHELLS

By

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ABSTRACT

(U) The authors review the basic principles of the closed quasilinear quadratic theory of viscoelasticity of physically nonlinear media as proposed by A. A. Il'yushin and P. M. Ogibalov. A system of nonlinear equations of bending and stability is proposed for flexible (i.e. with regard to geometric nonlinearity) shallow plates and shells made from orthotropic materials with linear properties (fiberglass-reinforced plastics). It is assumed that the hypothesis of straight normals is applicable to these plates and shells. It is also assumed that stresses normal to the middle surface are insignificantly small compared to the other components and that the shells and plates remain orthotropic throughout the entire deformation process. A method of solving the proposed equations is outlined and illustrated by analysis of the stability of a rectangular orthotropic plate of slightly curved panel of fiberglass-reinforced plastic with given stress relaxation curves. The results agree satisfactorily with experimental data on creep in a square plate hinged at the edges. Methods are also given for determining the upper and lower critical loads as related to the loading conditions and the critical time. A viscoelastic solution is found by the proposed method for the problem of stability of a compressed cylindrical shell and compared with an elastic solution found by the Ritz method. Orig. art. has: 36 formulas.

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STABILITY OF ORTHOTROPIC VISCOELASTIC SHELLS

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This article presents the fundamental concepts of heredity theories. For orthotropic materials with linear properties the article presents a system of nonlinear equations of flexure and stability for plates and shells, as well as a method for constructing solutions. Methods are presented for determining the upper and lower critical loads as a function of the loading regime and of the critical time.

The ever-increasing use of plastics, fiberglass and other synthetic materials requires providing new methods for solving engineering problems which take into account the rheonomy of their properties.

The elastic aftereffect phenomenon, discovered in 1834 by L. Vicat and then by K. Weber, was subjected in the middle of the past century to the scrutiny of such major scientists as F. Kohlrausch, R. Clausius, D. Maxwell, S. Thomson, L. Boltzmann, and others. Bringing his theory in line with experimental results L. Boltzmann put forward two hypotheses: 1) the stress depends not only on the strain prevailing at the given time, but also on the preceding strains, whose effect is the weaker, the farther they are removed in time from the present; 2) the law of independent force action is applicable to elastic forces.

On the basis of these hypotheses we shall represent the relationships between stresses and strains in time suggested by L. Boltzmann in the following form

$$\varepsilon(t) = \frac{\sigma(t)}{E} + \frac{1}{E}\int_0^t K(t-s)\,\sigma(s)\,ds, \qquad \sigma(t) = E\varepsilon(t) - E\int_0^t \Gamma(t-s)\,\varepsilon(s)\,ds. \tag{1}$$

In the beginning of the present century W. Volterra developed a theory of integral equations from which, in particular, follows a relationship between the kernel K and the resolvent Γ

$$K(t) - \Gamma(t) = \int_0^t \Gamma(t-s)\,K(s)\,ds. \tag{2}$$

Some nonlinear effects of creep for a noncompressible medium can be described by the equations

$$\varepsilon(t) = \psi(\sigma) + \int_0^t K(t-s)\,\psi[\sigma(s)]\,ds, \qquad \varphi(\varepsilon) = \sigma(t) + \int_0^t K(t-s)\,\sigma(s)\,ds. \tag{3}$$

where function $\psi(\sigma)$ is constructed from similarity of the creep curves, while function $\varphi(\varepsilon)$ is constructed from similarity of isochronous curves, characterizing the strain on rapid loading.

Reference [1] recommends to represent the general relationship $\sigma \sim \varepsilon \sim t$ between the stresses, strains and time by means of a linear tensor operator. References [2]-[4] suggest a closed quasi-linear quadratic theory of viscoelasticity of media which have a physical nonlinearity. Here are already established all the general principal one-to-one relationships between tensors σ_{ij}, ε_{ij} and time, and relationships are obtained between the secondary kernels of creep and relaxation, as well as between them and the corresponding kernels of the linear theory.

Let us consider the main concepts of the theory being suggested. Let $S_{ij}(t_0)$, $S_{ij}(t_1)$, ..., $S_{ij}(t_k)$, ..., $S_{ij}(t_N)$ be the stresses at times t_0, t_1, ..., t_k, ..., t_N of the interval $[t_0, t]$ (t being the time at which it is required to determine the stress in the body, acting during a short time interval Δt. Let us assume that it is required to study the function of process Z_{ij} at time t, for example, the strain which is produced by the entire ensemble of stress pulses $S_{ij}(t_k)$. The sum of contribution of individual stress pulses with the corresponding influence function will give a representation of Z_{ij} in the form of a linear operator of S_{ij} of the type of (1), which is the first approximation. The following approximation will be the combined action of two preceding stress pulses S_{ij} at times t_p and t_γ. This effect will be proportional to the product of these pulses during the applicable times, multiplied by the joint effect function of this pair. The following approximation is the effect of three stress pulses, acting during three different times, with the joint effect function of these three stresses, etc. For deviators of the stress and deformation tensors the relationship $\sigma \sim \varepsilon \sim t$ should be odd, since for a nonvarying value of $3\sigma = \sigma_{ii}\delta_{ii}$ a reversal in the direction of shear should change the sign of S_{ij}.

According to the isotropy postulate, the relationship $\sigma \sim \varepsilon \sim t$ should contain a triple product, which yields the quasi-linear tensor $S_{ij}(t_k)S_{mn}(t_p)S_{mn}(t_\gamma)$. As a result, for a process Z_{ij} (strains, for example) we get

$$Z_{ij}(t) = \int_0^t K(t,\tau)\,S_{ij}(\tau)\,d\tau + \int_0^t S_{ij}(\tau)\,d\tau \int_0^t\int_0^t K_3(t,\tau;\xi,\eta)\,S(\xi,\eta)\,d\eta\,d\xi. \tag{4}$$

The solution of this system of integral equations for stresses S_{ij} is

$$S_{ij}(T) = \int_t^T \Gamma(T-t)\,Z_{ij}(t)\,dt + \int_0^t Z_{ij}(t)\,dt \int_0^T\int_t^t \Gamma_3(T,t,\xi,\eta)\,Z(\xi,\eta)\,d\xi\,d\eta. \tag{5}$$

Equations (4) and (5) express the property of reciprocity: kernels K, Γ, K_3 and Γ_3 should be interrelated by integral equations which are not a function of Z and S. The

properties of resolvent Γ_3 should be analogous to those of kernel K_3, in particular, Γ_3 should be symmetrical with respect to arguments ξ and η.

The integral equation relating kernels K and K_3 and resolvents Γ and Γ_3 reduces to the form

$$\int_0^t K(t,\tau)\Gamma_3(\tau,\tau',p,q)\,d\tau = -\int_{\tau}^{t}\Gamma(\tau,\tau')\,d\tau\int_{\xi}\Gamma(\xi,p)\,d\xi\int_{\eta}\Gamma(\eta,q)\,K_3(t,\tau,\xi,\eta)\,d\eta. \quad (6)$$

The solution of this equation is expressed as

$$-\Gamma_3(t,\tau;\xi,\eta) = \int_{\max(\xi,\eta)}^{t}\Gamma(t,x)\,dx\int_{\tau}^{t}\Gamma(y,\tau)\,dy\int_{\xi}^{t}\Gamma(u,\xi)\,du\int_{\eta}^{t}\Gamma(v,\eta)\,K_3(x,y,u,v)\,dv. \quad (7)$$

Due to symmetry of kernels K and Γ it is possible to write an expression for K_3 in terms of Γ_3

$$-K_3(t,\tau;\xi,\eta) = \int_{\max(\xi,\eta)}^{t} K(t,x)\,dx\int_{\tau}^{t}K(y,\tau)\,dy\int_{\xi}^{t}K(u,\xi)\,du\int_{\eta}^{t}K(v,\eta)\,\Gamma_3(x,y;u,v)\,dv \quad (8)$$

It is clear from this that, the closer times t_k, t_l and t_p to time t, the greater is the effect on the process at hand of the values of $S_{ij}(t_m)$ and, consequently, this property shows that the expressions for K, Γ, K_3 and Γ_3 contain Dirac's δ-functions

$$\widetilde{K}(t,x) = \delta(t-x) + K(t,x); \qquad \widetilde{\Gamma}(t,x) = \delta(t-x) - \Gamma(t,x), \quad (9)$$

where K and Γ are regular kernels (for example, such as $\Gamma(t) = me^{-mt}$).

The general form of singular kernel $\widetilde{R}$ (which also means of $\widetilde{K}_3$ and $\widetilde{\Gamma}_3$), which is a scalar, has the form

$$\widetilde{R}(x) = R(x) + R_{ij}\beta_{ij}(x) + R_{ij,kl}\beta_{ij}(x)\beta_{kl}(x) + R_{ij,kl,mn}\beta_{ij}(x)\beta_{kl}(x)\beta_{mn}(x).$$

Here

$$\beta_{ij} = \begin{cases}\delta(x_i) & i = j,\\ \delta(x_{ij}) & i \neq j;\end{cases}$$

the quantity R with its subscripts forms a set of three "tensors" of second, fourth and sixth-order regular kernels, which are a function of x. Expressions of $\widetilde{K}_3$ and $\widetilde{\Gamma}_3$ in terms of δ-functions and regular kernels are more complex and are not presented (the reader is referred for them to [3]).

In constructing a theory of plates and shells made from material with rheonomic properties, it is first necessary to assume relationships in which the physical law is the first approximation of the general theory, given for the Boltzmann-Volterra linear processes in the form of (1) and (2). Here kernel K may be selected either with a singularity at $t = 0$, or as a sum of some $D(t, \alpha)$ kernel [6], which reflects the start of the process, and a regular K_1 in such a manner that the relationship $\sigma \sim \varepsilon \sim t$ will be

written in the form

$$\sigma(t) = E\left[\varepsilon(t) - \lambda\int_0^t D(t-s)\,\varepsilon(s)\,ds - \int_0^t K_1(t-s)\,\varepsilon(s)\,ds\right].$$

If the temperature shear function $a_{temp}(T) = t/t_{loc}$, where t is the time of observation and t_{loc} is some reduced or "local" time, for the material under study is known, then the principal relationships of the heredity theories, which are valid for any temperatures from the range of operation of temperature-time analogy, will be written by replacing time t by "time" t_{loc} [2, 5]. For example, in the linear case we will have

$$E\varepsilon = \int_0^t K_{temp}(t_{loc} - s_{loc})\,d\sigma(s); \qquad K_{temp}(t) = \int_0^t K(x)\,dx.$$

Let us now pass on to construction of equations of viscoelastic shells, understanding that R and K denote either a sum in the form $\lambda D + K_1$, or kernels with a singularity in $t = 0$.

Since plates and shells are made quite extensively from fiberglass, whose properties in the majority of cases are orthotropic, we shall construct the equations of the theory of shells for flexible (i.e., with consideration of geometric nonlinearity) orthotropic shallow shells, whose material displays linear heredity. We assume that the hypothesis of undeformable normals is applicable to fiberglass plates and shells. It is assumed in addition that the stresses acting normal to the middle surface are negligible in comparison with other components and that orthotropy is retained during the entire deformation process.

We select a coordinate system (x, y, z) lining up axes ox and oy along the base and weft (principal system), while axis oz is directed normal to the (x, y) coordinate plane. We have the following expressions for the deformation of the middle layer of the shell and for the curvatures in terms of the displacement of the middle layer:

$$\begin{aligned} e_{11}^0 &= \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - k_1 w, & \varkappa_x &= -\frac{\partial^2 w}{\partial x^2};\\ e_{22}^0 &= \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - k_2 w, & \varkappa_y &= -\frac{\partial^2 w}{\partial y^2};\\ e_{12}^0 &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}; & \varkappa_{xy} &= -\frac{\partial^2 w}{\partial x\partial y}. \end{aligned} \quad (10)$$

By virtue of the Kirchhoff theorem, the displacements and strains in a layer situated at distance z from the middle surface are

$$u = u_0 - z\frac{\partial w}{\partial x}; \qquad v = v_0 - z\frac{\partial w}{\partial y};$$
$$e_{11} = e_{11}^0 - z\frac{\partial^2 w}{\partial x^2}, \quad e_{22} = e_{22}^0 - z\frac{\partial^2 w}{\partial y^2}; \quad e_{12} = e_{12}^0 - 2z\frac{\partial^2 w}{\partial x\partial y} \quad (11)$$

The relationship between stresses and strains for orthotropic materials is written in the form

$$\sigma_{11} = B_{11}e_{11} - \qquad -\tau)e_{11}(\tau)d\tau + B_{12}e_{22} - \int_0^t R_{12}(t-\tau)e_{22}(\tau)d\tau;$$

$$\sigma_{22} = B_{22}\ldots - \int_0^t R_{21}(t-\tau)e_{11}(\tau)d\tau + B_{22}e_{22} - \int_0^t R_{22}(t-\tau)e_{22}(\tau)d\tau; \tag{12}$$

$$\sigma_{12} = 2Be_{12} - 2\int_0^t R'(t-\tau)e_{12}(\tau)d\tau.$$

Here

$$B_{11} = \frac{E_1}{1-\nu_1\nu_2}; \quad B_{12} = B_{21} = \frac{\nu_1 E_2}{1-\nu_1\nu_2} = \frac{\nu_2 E_1}{1-\nu_1\nu_2}; \quad B_{22} = \frac{E_2}{1-\nu_1\nu_2};$$

$$2B = G = \frac{E_{\pi/4}}{2(1+\nu_{\pi/4})}; \quad \nu_1 = \nu_{21} = \frac{e_{22}}{e_{11}}; \quad \nu_2 = \nu_{12} = \left|\frac{e_{11}}{e_{22}}\right| \tag{13}$$

are the components of the tensor of the moduli of elastic anisotropy and the Poisson ratios for tension along the base and along the weft;

$$R_{11} = B_{11}R_{11}(t); \quad R_{22} = B_{22}R_{22}(t); \quad R_{12} = B_{12}R_{12}(t); \quad R' = BR \tag{14}$$

are the components of the tensor of the relaxation kernels.

The forces acting per unit width of shell-element cross section are

$$T_k = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{kk}dz; \quad M_k = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{kk}zdz; \quad \widetilde{S} = \widetilde{S}_1 = \widetilde{S}_2 = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{12}dz.$$

$$H = H_1 = H_2 = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{12}zdz.$$

while stresses σ_{mn} are given by Eqs. (12). Substituting Eqs. (10), (12) and (14) into the expression for forces, we get

$$T_1 = hB_{11}(e^0_{11} - J_{11}e^0_{11}) + hB_{12}(e^0_{12} - J_{12}e^0_{22});$$
$$T_2 = hB_{12}(e^0_{11} - J_{12}e^0_{11}) + hB_{22}(e^0_{22} - J_{22}e^0_{22}); \tag{15}$$
$$\widetilde{S} = 2hB(e^0_{12} - Je^0_{12}).$$

where

$$J_{ij}e^0_{ii} = \int_0^t R_{ij}(t-\tau)e^0_{ii}(\tau)d\tau$$

After a Laplace transformation, Eqs. (15) take on the form

$$T_1^* = hB^*_{11}e^{0*}_{11} + hB^*_{12}e^{0*}_{22}, \quad T_2^* = hB^*_{12}e^{0*}_{11} + hB^*_{22}e^{0*}_{22};$$
$$\widetilde{S}^* = 2hB^*e^{0*}_{12} \tag{16}$$

Here

$$B^*_{ij} = B_{ij}(1 - R^*_{ij}); \quad R^*_{ij} = \int_0^\infty R_{ij}(t)e^{-pt}dt.$$

From Eqs. (16) we find

$$e^{0*}_{11} = \frac{1}{h}\frac{T_1^*B^*_{22} - T_2^*B^*_{12}}{B^*_{11}B^*_{22} - B^*_{12}B^*_{21}}, \quad e^{0*}_{22} = \frac{1}{h}\frac{B^*_{11}T_2^* - B^*_{21}T_1^*}{B^*_{11}B^*_{22} - B^*_{12}B^*_{21}}; \quad e^{0*}_{12} = \frac{\widetilde{S}^*}{2hB^*}$$

We set up the identity

$$\frac{\partial^2}{\partial y^2}[2B^*(B^*_{22}T_1^* - B^*_{12}T_2^*)] + \frac{\partial^2}{\partial x^2}[2B^*(B^*_{11}T_2^* - B^*_{21}T_1^*)] - \frac{\partial^2}{\partial x\partial y}(B^*_{11}B^*_{22} - B^*_{12}B^*_{21})\widetilde{S}^* =$$
$$= 2hB^*(B^*_{11}B^*_{22} - B^*_{12}B^*_{21})\left\{\frac{\partial^2}{\partial y^2}(e^{0*}_{11}) + \frac{\partial^2}{\partial x^2}(e^{0*}_{22}) - \frac{\partial^2}{\partial x\partial y}(e^{0*}_{12})\right\}, \tag{17}$$

into which we substitute expressions for e^{0*}_{ij} from the preceding expressions, we simplify and introduce the stress function φ using the formulas

$$\frac{\partial^2\varphi}{\partial y^2} = \frac{1}{h}T_1, \quad \frac{\partial^2\varphi}{\partial x^2} = \frac{1}{h}T_2; \quad \frac{\partial^2\varphi}{\partial x\partial y} = -\frac{1}{h}\widetilde{S},$$

and also apply the Laplace transform

$$T_1^* = h\frac{\partial^2\varphi^*}{\partial y^2}; \quad T_2^* = h\frac{\partial^2\varphi^*}{\partial x^2}, \quad \widetilde{S}^* = -h\frac{\partial^2\varphi^*}{\partial x\partial y}.$$

As a result we get a continuity equation [7] for an orthotropic elastic shell, whose material has linear heredity properties, in the operator form

$$B_{11}\frac{\partial^4\varphi^*}{\partial x^4} + \frac{1}{2B^*}(B^*_{11}B^*_{22} - B^*_{12}B^*_{21} - 4B^*B^*_{12})\frac{\partial^4\varphi^*}{\partial x^2\partial y^2} + B^*_{22}\frac{\partial^4\varphi^*}{\partial y^4} =$$
$$= (B^*_{11}B^*_{22} - B^*_{12}B^*_{21})\left\{\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]^* - \left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)^* - k_1\frac{\partial^2 w^*}{\partial y^2} - k_2\frac{\partial^2 w^*}{\partial x^2}\right\} \tag{18}$$

The right-hand side of the above expression has nonlinear terms $\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]^* \ldots\ldots$

It will be subsequently assumed that it is possible to represent the deflection in the form w(x, y, t) = w(x, y)w_1(t), where function w_1(t), its derivatives and their squares allow the Laplace transform.

With the aid of (12), the expressions for the moments have the form

$$M_1 = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{11}zdz = -\frac{h^3}{12}\left\{B_{11}\left[\frac{\partial^2 w}{\partial x^2} - \int_0^t R_{11}(t-\tau)\frac{\partial^2 w}{\partial x^2}d\tau\right] + \right. \tag{19}$$
$$\left. + B_{12}\left[\frac{\partial^2 w}{\partial y^2} - \int_0^t R_{12}(t-\tau)\frac{\partial^2 w}{\partial y^2}d\tau\right]\right\};$$

$$M_2 = -\frac{h^3}{12}\left\{B_{21}\left[\frac{\partial^2 w}{\partial x^2} - \int_0^t R_{21}(t-\tau)\frac{\partial^2 w}{\partial x^2}d\tau\right] + B_{22}\left[\frac{\partial^2 w}{\partial y^2} - \int_0^t R_{22}(t-\tau)\frac{\partial^2 w}{\partial y^2}d\tau\right]\right\},$$

$$H = -\frac{h^3}{12}B\left[\frac{\partial^2 w}{\partial x \partial y} - \int_0^t R(t-\tau)\frac{\partial^2 w}{\partial x \partial y}d\tau\right]. \tag{19}$$

Introducing the values of M_1, M_2, H and of T_1, T_2, $\tilde{S}$ into the equilibrium equation of the shell

$$\frac{\partial^2 M_1}{\partial x^2} + 2\frac{\partial^2 H}{\partial x \partial y} + \frac{\partial^2 M_2}{\partial y^2} + T_1\left(k_1 + \frac{\partial^2 w}{\partial x^2}\right) + T_2\left(k_2 + \frac{\partial^2 w}{\partial y^2}\right) + 2\tilde{S}\frac{\partial^2 w}{\partial x \partial y} + q = 0$$

and applying the Laplace transform, we find the equilibrium equation for an elastic orthotropic shell from a material with linear heredity properties

$$B_{11}^*\frac{\partial^4 w^*}{\partial x^4} + 2(B_{12}^* + 4B^*)\frac{\partial^4 w}{\partial x^2 \partial y^2} + B_{22}^*\frac{\partial^4 w^*}{\partial y^4} - \frac{12}{h^3}\left(k_2\frac{\partial^2 \varphi^*}{\partial x^2} + k_1\frac{\partial^2 \varphi^*}{\partial y^2}\right) -$$
$$-\frac{12}{h^3}\left[\left(\frac{\partial^2 \varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)^* + \left(\frac{\partial^2 \varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right)^* - 2\left(\frac{\partial^2 \varphi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right)^*\right] - \frac{12}{h^3}q^* = 0. \tag{20}$$

Here $B_{ij}^* = B_{ij}(1 - R_{ij}^*)$ is a known function of p.

In intermediate calculations we have obtained the expressions

$$M_1^* = -\frac{h^3}{12}\left(B_{11}^*\frac{\partial^2 w}{\partial x^2} + B_{12}^*\frac{\partial^2 w^*}{\partial y^2}\right);$$

$$M_2^* = -\frac{h^3}{12}\left(B_{21}^*\frac{\partial^2 w^*}{\partial x^2} + B_{22}^*\frac{\partial^2 w^*}{\partial y^2}\right); \tag{21}$$

$$H^* = -\frac{h^3}{3}B^*\frac{\partial^2 w^*}{\partial x \partial y}.$$

We thus have two resolving equations (18) and (20), for the unknown stress (x, y, t) and deflection w(x, y, t) functions. We note that these two equations contain terms of two types: such as $B_{11}^*\frac{\partial^4 w^*}{\partial x^4}$, in which coefficients B_{ij}^* are known function of the complex parameter p, while the transform of the unknown functions is contained in the linear form, and such as $\left(\frac{\partial^2 \varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)^*$, where, by virtue of the method being presented, one of the functions (for example, w) contained in parentheses will be considered as a known function of position and time, specified to within undeterminate constants. Consequently, with respect to unknowns not specified by functions (for example, φ) the above product will be linear.

To clarify this method we now present one of the possible approximate solutions of the system of equations (18) and (20). We shall consider an orthotropic plate or weakly bent panel, rectangular in the plan, from fiberglass, for which the stress relaxation curves in specimens under tension are known and are representable, for example, in the form

$$\sigma(t) = \sigma_0[1 - \gamma(1 - e^{-\alpha t})]. \tag{22}$$

Obviously, the kernel of relaxation will be

$$R = \gamma\alpha e^{-\alpha t}, \qquad (0 < \gamma < 1,\ 0 < \alpha < 1), \tag{23}$$

where α and γ depend on the orientation of the specimen.

The transform of the kernel has the form

$$R^* = \frac{\gamma\alpha}{p + \alpha}. \tag{24}$$

We assume henceforth for simplicity that all the

$$B_{ik}^* = B_{ik}(1 - R^*) \tag{25}$$

and that w, the deflection function is given in the form

$$w(x, y, t) = w_0(x, y)\,w_1(t), \tag{26}$$

where w_0(x, y) is the elastic solution, while function w_1(t) is selected to within unknown parameters λ and μ in the form

$$w_1(t) = \lambda - \mu e^{-\alpha t}, \qquad \lambda - \mu = 1. \tag{27}$$

Stress function φ is sought in the form

$$\varphi(x, y, t) = \varphi_0(x, y)\,\varphi_1(t), \tag{28}$$

here φ_0(x, y) is the elastic solution, while φ_1(t) for a given form of function w_1(t) is to be determined.

Laplace transform of w_1(t) yields

$$w_1^*(p) = \frac{\lambda}{p} - \frac{\mu}{p + \alpha}. \tag{29}$$

Substituting Eqs. (25), (26) and (28) into continuity equation (13), we find

$$\varphi_1^* = \varkappa_1\delta(1 - R^*)w_1^* + n_1\delta(1 - R^*)(w_1^2)^*. \tag{30}$$

Here

$$\varkappa_1 = -\frac{\nabla_k^2 w_0}{\nabla^2\nabla^2\varphi_0} = \frac{-\left(k_1\frac{\partial^2 w_0}{\partial y^2} + k_2\frac{\partial^2 w_0}{\partial x^2}\right)}{B_{11}\frac{\partial^4 \varphi_0}{\partial x^4} + C\frac{\partial^4 \varphi_0}{\partial x^2 \partial y^2} + B_{22}\frac{\partial^4 \varphi_0}{\partial y^4}};$$

$$C = \frac{1}{2B}(\delta - 4BB_{12}), \qquad \delta = B_{11}B_{22} - B_{12}^2;$$

$$n_1 = \frac{N_0}{\nabla^2\nabla^2\varphi_0} = \frac{\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2 - \frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}}{\nabla^2\nabla^2\varphi_0}.$$

Substituting Eqs. (24) and (29) into Eq. (30) and having reference to the fact that

$$\frac{1}{(p+k\alpha)(p+m\alpha)} = \frac{1}{(m-k)\alpha(p+k\alpha)} - \frac{1}{(m-k)\alpha(p+m\alpha)},$$

we get

$$\varphi_1^* = \frac{A_1}{p} + \frac{A_2}{p+\alpha} + \frac{A_3}{p+2\alpha} + \frac{A_4}{(p+\alpha)^2}. \quad (31)$$

Inverting (31) we find

$$\varphi_1 = A_1 + A_2 e^{-\alpha t} + A_3 e^{-2\alpha t} + A_4 t e^{-\alpha t}, \quad (32)$$

where

$$A_1 = \delta(1-\gamma)(\lambda \varkappa_1 + \lambda^2 n_1); \quad A_2 = n_1 \delta(\gamma\lambda^2 - 2\lambda\mu - \gamma\mu^2) - \varkappa_1 \delta(\mu - \lambda\gamma).$$
$$A_3 = n_1 \delta \mu^2 (1+\gamma); \quad A_4 = \delta\gamma\alpha(\varkappa_1\mu + 2n_1\lambda\mu). \quad (33)$$

To determine the sought parameters λ and μ we substitute Eqs. (26) and (28), with reference to (27) and (32), into equilibrium equation (20). Then

$$\frac{K_1}{p} + \frac{K_2}{p+\alpha} + \frac{K_3}{p+2\alpha} + \frac{K_4}{p+3\alpha} + \frac{K_5}{(p+\alpha)^2} + \frac{K_6}{(p+2\alpha)^2} = 0.$$

which, after inversion, yields an equation relating the load and the deflection in the form

$$K_1 + K_2 e^{-\alpha t} + K_3 e^{-2\alpha t} + K_4 e^{-3\alpha t} + K_5 t e^{-\alpha t} + K_6 t e^{-2\alpha t} = 0. \quad (34)$$

Here

$$K_1 = \nabla^2\nabla^2 \tilde{w}_0 \lambda(1-\gamma) - \frac{12}{h^2}\nabla_k^2\varphi_0 A_1 - \frac{12}{h^2} M_0 \lambda A_1 - \frac{12}{h^3} q(t);$$

$$K_2 = -\nabla^2\nabla^2 \tilde{w}_0(\mu - \lambda\gamma) - \frac{12}{h^2}\nabla_k^2\varphi_0 A_2 - \frac{12}{h^2} M(\lambda A_2 - \mu A_1);$$

$$K_3 = -\frac{12}{h^2}\nabla_k^2\varphi_0 A_3 - \frac{12}{h^2} M_0(\lambda A_3 - \mu A_2); \quad K_4 = \frac{12}{h^2} M_0 \mu A_3;$$

$$K_5 = \nabla^2\nabla^2 \tilde{w}_0 \gamma\alpha\mu - \frac{12}{h^2}\nabla_k^2\varphi A_4 - \frac{12}{h^2}\lambda A_4 M_0, \quad K = \frac{12}{h^2} M_0 \mu A_4$$

From Eq. (34) we get

$$K_1 + K_2 + K_3 + K_4 = 0 \text{ when } t = 0.$$
$$K_1 = 0 \text{ when } t = \infty. \quad (35)$$

To determine λ we substitute the value of A_1 from Eq. (33) into (35). As a result we get, for example, for $q = q_0 = \text{const}$

$$\lambda^3 - (\varkappa_2 - \varkappa_1)\lambda^2 - \left(\frac{h^2}{12\delta}\frac{m_1}{n_1} + \varkappa_1\varkappa_2\right)\lambda + \frac{q_0 m}{k(1-\gamma)} = 0. \quad (36)$$

Here

$$\varkappa_1 = \frac{\nabla_k^2 w_0}{N_0}; \quad \varkappa_2 = \frac{\nabla_k^2 \varphi}{M_0}; \quad m_1 = \frac{\nabla^2\nabla^2 \tilde{w}_0}{M_0}; \quad n_1 = \frac{N_0}{\nabla^2\nabla^2\varphi_0};$$

$$m = \frac{\nabla^2\nabla^2\varphi_0}{\delta N_0 M_0}; \quad M_0 = \frac{\partial^2\varphi_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial^2\varphi_0}{\partial y^2}\frac{\partial^2 w_0}{\partial x^2} - 2\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial^2\varphi_0}{\partial x\partial y}.$$

Solution of Eq. (36) for a square hinged plate from orthotropic polyester fiberglass with the following components of the tensor of elastic moduli B_{ij}

$$B_{11} = 1.49\cdot 9.81\cdot 10^9 \text{ N/m}^2; \quad B_{22} = 1.143\cdot 9.81\cdot 10^9 \text{ N/m}^2;$$
$$B_{12} = B_{21} = 0.18\cdot 9.81\cdot 10^9 \text{ N/m}^2; \quad B = 0.13\cdot 9.81\cdot 10^9 \text{ N/m}^2$$

yields one real root $\lambda = 1.21$, which is in good agreement with experimental data on the creep of a square plate with hinged edges ($\lambda = 1.19$).

In solving problems of shell stability it is useful to select function $w_1(t)$ in the form

$$w_1(t) = \lambda - \mu e^{-\alpha t} - \beta\vartheta(t - t_{cr}),$$

here

$$\vartheta = \begin{cases} 0 & t < t_{cr}, \\ 1 & t = t_{cr}. \end{cases}$$

The "popping" parameter β will be found by equating the "jump" in the deflection of the viscoelastic problem, to the deflection "jump" in the elastic problem, since it may be assumed that creep will not develop during the popping.

We now present the results from solving the problem of stability of a compressed circular cylindrical shell from PN-1, T-1 polyester fiberglass along the axis; here the weft of the fiber is directed along the generatrix, while the base is directed along the directrix.

Creep and relaxation experiments show that it is possible to assume

$$\gamma_{11} = \gamma_{22} = \frac{1}{6}; \quad \alpha_{11} = \alpha_{12} = \alpha_{22} = 0.5; \quad \gamma_{12} = \mu_1\gamma_{11}, \quad \gamma = 3\gamma_{11},$$

where μ_1 is of the order of magnitude of the Poisson ratio and is to be determined (it was found from conditions at $t = \infty$ that $\mu = 0.12$) the elastic solution of the problem was found by the Ritz method, while the viscoelastic solution was obtained by the method presented above.

It follows from the solution for a constant load that the critical loads are the same as in the elastic case, while the critical deflections increase; for a time-varying load the critical loads become smaller, while the deflections become larger as compared

with the elastic values. Thus, $\hat{P}_{up.el} = 0.558$, $\hat{P}_{up.v.el} = 0.483$, $\hat{P}_{l.el} = 0.171$, $\hat{P}_{l.v.el} = 0.137$. We note that the values of the "upper" and "lower" critical loads for viscoelastic shells depends appreciably on the loading rate (with an increase in this rate the "upper" critical load becomes greater).

We note an important circumstance which can be successfully used, namely: if the temperature range for which the temperature-time analogy $T \sim t$ holds is known, i.e., the temperature shear function $a_{temp}(T) = t/t_{loc}$ for the given material has been constructed, then for any temperature T_k from this range the solutions are obtained by simple replacement of the time axis t by the "local" time $T_{loc} = a_{temp}t$, where T_k is the shell's "functioning" temperature.

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